FIBRATIONS, DIVISORS AND TRANSCENDENTAL LEAVES

JORGE VITÓRIO PEREIRA

(WITH AN APPENDIX BY LAURENT MEERSSEMAN)

ABSTRACT. We will use flat divisors, and canonically associated singular holomorphic foliations, to investigate some of the geometry of compact complex manifolds. The paper is mainly concerned with three distinct problems: the existence of fibrations, the topology of smooth hypersurfaces and the topological closure of transcendental leaves of foliations.

CONTENTS

1. INTRODUCTION AND STATEMENT OF RESULTS

Let M be a compact complex manifold and Div(M) be its group of divisors, i.e., the free abelian group generated be the irreducible hypersurfaces of M. We will denote by $\mathrm{Div}_{S^1}(M)$ the subgroup of $\mathrm{Div}(M)$ formed by S^1-flat divisors, i.e., divisors D whose associated line bundle $\mathcal{O}_M(D)$ admits a hermitian flat connection. We will denote by $\Gamma(M)$ the quotient of the group of rational divisors by the group of rational S^1-flat divisors, i.e.,

$$\Gamma(M) = \frac{\mathrm{Div(M)} \otimes \mathbb{Q}}{\mathrm{Div}_{S^1}(M) \otimes \mathbb{Q}}.$$

If M is a projective manifold then

$$\Gamma(M) = NS_{\mathbb{Q}}(M),$$

i.e., $\Gamma(M)$ can be identified with the rational Neron-Severi group of M, see section 4, and in particular is finite dimensional. Our first result says that this is always the case for compact complex manifolds, i.e.,

Theorem 1. *Let M be a compact complex manifold then $\dim_{\mathbb{Q}} \Gamma(M) < \infty$.*

Most of our results involve *the Γ-class of a divisor*, i.e., the image of the divisor under the natural homomorphism

$$\mathrm{Div(M)} \to \Gamma(M) \,.$$

Note that the Γ-class of a divisor completely determines its rational-Chern class or, equivalently, its rational homology class. For Kaehler varieties the converse also holds, i.e., the Γ-class of a divisor is completely determined by its rational Chern-class. In fact in this case $\Gamma(M)$ is isomorphic to the image of the group of divisors under the rational Chern class map.

¿From some basic properties of flat divisors and Theorem 1 we deduce a characterization of compact complex manifolds which fibers over a projective curve.

Theorem 2. *Let M be a compact complex manifold with $\dim_{\mathbb{Q}} \Gamma(M) + 2$ pairwise disjoint hypersurfaces H_i. Then there exists a holomorphic map $\rho : M \to C$, C a smooth algebraic curve, with connected fibers such that every H_i is a component of a fiber of ρ.*

This result was inspired by an unpublished result due to A. Vistoli stating that a smooth projective variety, over an arbitrary field, with an infinite number of disjoint hypersurfaces fibers over a curve. In the case of compact Kaehler varieties Vistoli's proof has been adapted by M. Sebastiani in [8].

A related result for projective varieties has been obtained by B. Totaro, see Theorem 2.1 of [9]. The proof of Theorem 2 also prove a generalization of Theorem 2.1 of [9] to arbitrary complex manifolds.

Theorem 3. *Let M be a compact complex variety. Let $D_1, D_2, \ldots, D_r$, $r \geq 3$, be connected effective divisors which are pairwise disjoint and whose Γ-classes lie in a line of $\Gamma(M)$. Then there exists a map $\rho : M \to C$ with connected fibers to a smooth curve C such that $D_1, \ldots, D_r$ which maps the divisors D_i to points.*

As we will see some real foliations of codimension one are naturally associated to flat divisors. The pertubation of these foliations allow us to prove the next result,

Theorem 4. *Let M be a compact complex manifold. If H_1 and H_2 are smooth connected disjoint hypersurfaces such that $[H_1]$ and $[H_2]$ lie in the same line of $\Gamma(M)$ then there exists an étale $\mathbb{Z}/n$-covering $\tilde{D}_1$ of D_1 and an étale $\mathbb{Z}/m$-covering $\tilde{D}_2$ of D_2 which are diffeomorphic where m and n are positive integers satisfying $m[H_1] = n[H_2]$.*

It has to be noted that when M and H_1 and H_2 are smooth connected hypersurfaces and the Picard variety of M is isogeneous to a product of elliptic curves then it is shown in [9] that there exists finite and cyclic étale coverings of H_1 and H_2 with the same pro-l homotopy type. Theorem 4 is a generalization of this result and gives a positive answer to a conjecture made by B. Totaro in the above mentioned paper.

In analogy with the projective case we will say that a leaf L of a codimension one holomorphic foliation of a compact complex manifold M is transcendental if it

is not contained in any compact complex hypersurface. Our last main result is the following

Theorem 5. *Let M be a compact complex manifold, $\mathcal{F}$ a codimension one holomorphic foliation of M and L a transcendental leaf of $\mathcal{F}$. Denote by $\mathcal{H}$ the set of compact complex irreducible hypersurfaces of M which do not intersect the topological closure of L. Then the following assertion holds*

(1) *In general the cardinality of $\mathcal{H}$ is at most $\dim_{\mathbb{Q}} \Gamma(M) + 1$ and when is equal to $\dim_{\mathbb{Q}} \Gamma(M) + 1$ then $\mathcal{F}$ is given by a closed meromorphic 1-form;*

(2) *If M is projective then the cardinality of $\mathcal{H}$ is at most $\dim_{\mathbb{Q}} \Gamma(M)$ and when is equal to $\dim_{\mathbb{Q}} \Gamma(M)$ then $h^1(M, \mathcal{O}_M) \neq 0$ and $\mathcal{F}$ is given by a closed meromorphic 1-form. In particular for projective manifolds without global holomorphic 1-forms we have that the cardinality of $\mathcal{H}$ is at most $\dim_{\mathbb{Q}} \Gamma(M) - 1$.*

We remark that we cannot replace in the statement of Theorem 5 item (1) the group $\Gamma(M)$ by the Neron-Severi group of M, see Section 8. In an appendix to this paper L. Meersseman constructs a complex manifold of dimension 5 showing that in Theorems 2, 3 and 4 it is also not possible to replace $\Gamma(M)$ by the Neron-Severi group of M.

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2. Flat line bundles over complex manifolds

If M is a complex manifold then the set of isomorphism classes of holomorphic line-bundles is identified with $H^1(M, \mathcal{O}_M^*)$ as follows: let $\mathcal{L}$ be a holomorphic line bundle, $\mathcal{U} = \{U_i\}$ a sufficiently fine open covering of M and $\phi_i : U_i \to \mathcal{L}$ be nowhere vanishing local holomorphic sections of $\mathcal{L}$; over $U_i \cap U_j$ we have the transition functions $\phi_{ij} = \phi_i \cdot \phi_j^{-1} : U_i \cap U_j \to \mathbb{C}^*$ satisfying the identities $\phi_{ij} \cdot \phi_{ji} = 1$ and $\phi_{ij} \cdot \phi_{jk} \cdot \phi_{ki} = 1$. Therefore the collection $\{\phi_{ij}\}$ determines and element of $H^1(M, \mathcal{O}_M^*)$. It can be verified that this element does not depend on the choices made above.

Denote by $\mathbb{C}^*$, resp. S^1, the constant sheaf over M of invertible complex numbers, resp. complex numbers of modulus 1.

Definition 2.1. A line bundle $\mathcal{L} \in H^1(M, \mathcal{O}_M^*)$ is $\mathbb{C}^*$-flat, resp. S^1-flat, if $\mathcal{L}$ belongs to the image of the morphism $H^1(M, \mathbb{C}^*) \to H^1(M, \mathcal{O}_M^*)$, resp. $H^1(M, S^1) \to H^1(M, \mathcal{O}_M^*)$, induced by the natural inclusions.

Concretely a line bundle bundle $\mathcal{L}$ is $\mathbb{C}^*$-flat, resp. S^1-flat, if it admits a system of local holomorphic sections whose transition functions are locally constant holomorphic functions, resp. locally constant holomorphic functions of modulus 1.

We recall that the *Chern class* of a line bundle $\mathcal{L}$, denoted by $c(\mathcal{L})$, is the image of $\mathcal{L}$ under the map $H^1(M, \mathcal{O}_M^*) \to H^2(M, \mathbb{Z})$ induced by the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_M \to \mathcal{O}_M^* \to 0 \,.$$

The *real Chern class* of $\mathcal{L}$, denoted by $c_{\mathbb{R}}(\mathcal{L})$, is the image of $c(\mathcal{L})$ under the natural map $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})$.

The relations between flat line bundles and line bundles with zero real Chern class are presented in the next proposition. Its content is quite standard but due to a lack of references we will sketch its proof.

Proposition 2.1. *If $\alpha : H^1(M,\mathbb{C}) \to H^1(M,\mathcal{O}_M)$ and $\beta : H^1(M,\mathbb{R}) \to H^1(M,\mathcal{O}_M)$ are the morphisms induced by the natural inclusions then the following assertions hold:*

(1) *if $\mathcal{L}$ is $\mathbb{C}^*$-flat then $c_{\mathbb{R}}(\mathcal{L}) = 0$;*
(2) *if α is surjective then $\mathcal{L}$ is $\mathbb{C}^*$-flat if, and only if, $c_{\mathbb{R}}(\mathcal{L}) = 0$;*
(3) *if β is surjective then $\mathcal{L}$ is S^1-flat if, and only if, $c_{\mathbb{R}}(\mathcal{L}) = 0$;*
(4) *if the image of α is equal to the image of β then $\mathcal{L}$ is $\mathbb{C}^*$-flat if, and only if, $\mathcal{L}$ is S^1-flat.*
(5) *if M is compact then β is injective and consequently β is an isomorphism if and only if $h^1(M,\mathcal{O}_M) = 2h^1(M,\mathbb{R})$.*
(6) *if M is compact then the morphism $H^1(M,S^1) \to H^1(M,\mathcal{O}_M^*)$, induced by the natural inclusion, is injective.*

Proof. Consider the commutative diagram of sheaves of abelian groups over M

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_M & \longrightarrow & \mathcal{O}_M^* & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^* & \longrightarrow & 0
\end{array}
$$

with exact rows. From it we obtain the commutative diagram

$$
\begin{array}{ccccccc}
H^1(M,\mathbb{R}) & \longrightarrow & H^1(M,S^1) & \longrightarrow & H^2(M,\mathbb{Z}) & \longrightarrow & H^2(M,\mathbb{R}) \\
\beta \downarrow & & \downarrow & & \downarrow & & \\
H^1(M,\mathcal{O}_M) & \longrightarrow & H^1(M,\mathcal{O}_M^*) & \longrightarrow & H^2(M,\mathbb{Z}) & & \\
\alpha \uparrow & & \uparrow & & \uparrow & & \\
H^1(M,\mathbb{C}) & \longrightarrow & H^1(M,\mathbb{C}^*) & \longrightarrow & H^2(M,\mathbb{Z}) & \longrightarrow & H^2(M,\mathbb{C})
\end{array}
$$

with exact rows.

The proof of the proposition will be a standard chasing on the diagram above.

If $\mathcal{L}$ is a $\mathbb{C}^*$-flat line-bundle then follows from the exactness of the bottom row of the diagram above that $c(\mathcal{L}) \in \ker\{H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})\}$. Since $\ker\{H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})\} = \ker\{H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})\}$ we have that $c_{\mathbb{R}}(\mathcal{L}) = 0$. This proves assertion (1).

If $\mathcal{L}$ is a line bundle such that $c_{\mathbb{R}}(\mathcal{L}) = 0$ then we infer from the diagram that there exists $\theta \in H^1(M,\mathbb{C}^*)$ such that $c(\mathcal{L} \otimes \theta) = 0 \in H^2(M,\mathbb{Z})$, in particular $\mathcal{L} \otimes \theta \in \mathrm{Im}\{H^1(M,\mathcal{O}_M) \to H^1(M,\mathcal{O}_M^*)\}$. If α is surjective it follows that $\mathcal{L} \in \mathrm{Im}\{H^1(M,\mathbb{C}^*) \to H^1(M,\mathcal{O}_M^*)\}$ proving that $\mathcal{L}$ is $\mathbb{C}^*$-flat. This proves assertion (2).

Assertions (3) and (4) follows from completely analogous arguments and assertions (5) and (6) follows from the fact that pluriharmonic functions over compact complex manifolds are constant. $\qquad\square$

As the proposition above suggests we do not have in general the equivalence between zero real Chern class, $\mathbb{C}^*$-flat and S^1-flat.

Example 2.1. If M is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by $(z, w) \mapsto (\lambda_1 \cdot z, \lambda_2 \cdot w)$ with $0 < |\lambda_1| \le |\lambda_2| < 1$ then $H^1(M, \mathcal{O}_M^*) = H^1(M, \mathbb{C}^*) = \mathbb{C}^*$ and $H^1(M, S^1) = S^1$, see [1, pg. 172]. In particular every line bundle over M is $\mathbb{C}^*$-flat and there exists line bundles over M which are not S^1-flat.

Example 2.2. There exist complex manifolds M, diffeomorphic to $S^3 \times S^3$, such that $H^2(M, \mathbb{Z}) = H^1(M, \mathbb{C}) = 0$ and $H^1(M, \mathcal{O}_M) \ne 0$. Over these manifolds every line bundle has zero real Chern class and a line bundle is $\mathbb{C}^*$-flat if, and only if, it is the trivial line bundle.

If M is a compact complex surface then α is always surjective, see [1, page 117]. Therefore it follows from item (3) of proposition 2.1 the next

Corollary 2.1. *On compact complex surfaces a line bundle is $\mathbb{C}^*$-flat if, and only if, it has zero real Chern class.*

If M is a compact complex Kaehler manifold then it follows from Hodge Theory that β is always surjective. In particular we have the following

Corollary 2.2. *On a compact complex Kaehler manifold a line bundle is S^1-flat if, and only if, it has zero real Chern class.*

3. Flat divisors on complex manifolds

A divisor D on a complex manifold M is a formal sum

$$D = \sum_{i=1}^{\infty} d_i D_i \,,$$

where $d_i \in \mathbb{Z}$ and $\{D_i\}_{i \in \mathbb{N}}$ is a locally finite sequence of irreducible hypersurfaces of M. If $\mathcal{U} = \{U_i\}$ a sufficiently fine open covering of M then given a divisor D we can associate a collection of meromorphic function $f_i : U_i \dashrightarrow \mathbb{P}^1$ such that the restriction of D to U_i coincides with the divisor $(f_i)_0 - (f_i)_\infty$. Since the functions f_i are unique up to multiplication by a nowhere vanishing holomorphic function over U_i we can identify the (additive) group of divisors on M, denoted by $Div(M)$, with the (multiplicative) group $H^0(M, \mathcal{M}_M^*/\mathcal{O}_M^*)$, where $\mathcal{M}_M^*$ denotes the sheaf of invertible meromorphic functions over M.

Looking at the long exact sequence in cohomology associated to the short exact sequence of abelian groups

$$0 \to \mathcal{O}_M^* \to \mathcal{M}_M^* \to \frac{\mathcal{M}_M^*}{\mathcal{O}_M^*} \to 0 \,,$$

we obtain a map $H^0(M, \mathcal{M}_M^*/\mathcal{O}_M^*) \to H^1(M, \mathcal{O}_M^*)$, i.e., we obtain a map from the group of divisors over M to the group of isomorphism classes of line bundles over M. As usual we will denote the image of a divisor D by $\mathcal{O}_M(D)$.

Definition 3.1. We will say that a divisor $D \in Div(M)$ is $\mathbb{C}^*$-flat, resp. S^1-flat, if $\mathcal{O}_M(D)$ is a $\mathbb{C}^*$-flat, resp. S^1-flat, line bundle.

A divisor D on a complex manifold M is said to be *linearly equivalent to zero* if $\mathcal{O}_M(D) \equiv \mathcal{O}_M$. Equivalently, there exits a meromorphic function $f : M \dashrightarrow \mathbb{P}^1$ such that $D = (f)_0 - (f)_\infty$. In particular D is $\mathbb{C}^*$-flat. Note that if we take ω the rational 1-form over $\mathbb{P}^1$ with simple poles at zero and infinity then $f^*\omega$ will be a closed meromorphic one-form over M with simple poles along the support of D and holomorphic on the complement. A similar property holds for a general $\mathbb{C}^*$-flat divisor as the next proposition shows.

Proposition 3.1. *If $D \neq 0$ is a $\mathbb{C}^*$-flat line bundle over a complex manifold M then there exists a closed meromorphic one-form ω with simple poles along the support of D and holomorphic on the complement.*

Proof. If $\{U_i\}$ is sufficiently fine covering of M and $f_i : U_i \dashrightarrow P^1_{\mathbb{C}}$ are meromorphic maps such that $(f_i)_0 - (f_i)_\infty = D_{|U_i}$ then an explicit description of $\mathcal{O}_M(D)$ is given by the transition functions $f_{ij} = f_i \cdot f_j^{-1} : U_i \cap U_j \to \mathbb{C}^*$. By hypothesis the cocycle $\{f_{ij}\}$ is cohomologous to a locally constant cocycle $\{t_{ij}\}$. Concretely there exists holomorphic functions $t_i : U_i \to \mathbb{C}^*$ such that $f_{ij} \cdot t_{ij}^{-1} = t_i \cdot t_j^{-1}$. Therefore the meromorphic functions $F_i : U_i \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$, $F_i = \frac{f_i}{t_i}$ satisfies

(1) $(F_i)_0 - (F_i)_\infty = D_{|U_i}$
(2) $F_i = t_{ij} \cdot F_j$.

In particular, over $U_i \cap U_j$, we obtain the equality

$$\frac{dF_i}{F_i} = \frac{dF_j}{F_j}\,,$$

which implies that there exists a closed meromorphic 1-form ω such that

$$\omega_{|U_i} = \frac{dF_i}{F_i}\,,$$

for every open set $U_i \in \mathcal{U}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is important to note that in general the 1-form ω constructed above is not unique. In fact for two distinct choices of flat local equations for D we obtain two meromorphic 1-forms with simple poles along D differing by a global closed holomorphic 1-form η.

Reciprocally if $\{F_i : U_i \dashrightarrow \mathbb{P}^1_{\mathbb{C}}\}$ is a collection of flat local equations for D as in the proof of proposition 3.1 and η is a closed holomorphic 1-form over M then choosing arbitrary branches of $H_i = \exp \int \eta$ over each open set U_i we obtain a new collection of flat local equations $G_i = F_i/H_i$. Moreover we have that over each open set U_i the equality

$$\frac{dG_i}{G_i} = \frac{dF_i}{F_i} - \eta\,.$$

In other words, if $\Omega^1_{M,\mathrm{closed}}$ denotes the sheaf of closed holomorphic 1-forms over M then from the short exact sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{O}_M^* \xrightarrow{\;d\log\;} \Omega^1_{M,\mathrm{closed}} \longrightarrow 0$$

one deduces that the kernel of the map $H^1(M, \mathbb{C}^*) \to H^1(M, \mathcal{O}_M^*)$ coincides with the image of the map $H^0(M, \Omega^1_{M,\mathrm{closed}}) \to H^1(M, \mathbb{C}^*)$.

Example 3.1. Let M be, as in example 2.1, the quotient of $\mathbb{C}^2 \setminus \{0\}$ by $(z, w) \mapsto (\lambda_1 \cdot z, \lambda_2 \cdot w)$ with $0 < |\lambda_1| \leq |\lambda_2| < 1$. Suppose further that $\lambda_1^k \neq \lambda_2^l$ for every

$(k, l) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Under this hypothesis M contains just two irreducible curves; they are elliptic curves corresponding to the quotient of the axis by the contraction above, see [1, pg. 173]. Since every line bundle over M is $\mathbb{C}^*$-flat the same is true for divisors. Moreover there is no divisor D, $D \neq 0$, on M linearly equivalent to zero.

If we restrict to S^1-flat divisors we can refine proposition 3.1 to obtain the stronger

Proposition 3.2. *If $D \neq 0$ is a S^1-flat line bundle over a complex manifold M then there exists a closed meromorphic one-form ω_D with simple poles along the support of D and holomorphic on the complement. Moreover,*

(1) *If H is a compact hypersurface which does not intersect the support of D then $i^*\omega_D = 0$, where $i : H \to M$ denotes the natural inclusion*

(2) *If M is compact the meromorphic 1-form ω_D is unique up to a multiplicative constant.*

Proof. As in the proof of proposition 3.1, and using the same notation, we can construct meromorphic functions $F_i : U_i \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ satisfying

(1) $(F_i)_0 - (F_i)_\infty = D_{|U_i}$;

(2) $F_i = t_{ij} \cdot F_j$;

where t_{ij} are locally constant functions. The difference is that now the functions t_{ij} have modulus 1.

We define ω_D by the relations

$$\omega_{D|U_i} = \frac{dF_i}{F_i},$$

for every open set $U_i \in \mathcal{U}$.

If H is a hypersurface disjoint from the support of D we have that the line bundle $\mathcal{O}_M(D)$ is trivial when restricted to H. Moreover if H is compact then the map $H^1(H, S^1) \to H^1(H, \mathcal{O}_H^*)$ is injective, see item (6) of proposition 2.1, and it follows that the functions $F_{i|U_i \cap H}$ are constant whenever $U_i \cap H \neq \emptyset$. Thus if $i : H \to M$ denotes the inclusion then $i^*\omega = 0$. This proves assertion (1).

Assertion (2) follows from similar considerations. $\square$

Definition 3.2. *If $D \neq 0$ is a S^1-flat divisor of a compact complex manifold M then $\mathcal{F}_D$ is the codimension one singular holomorphic foliation induced by ω_D.*

Note that the foliation $\mathcal{F}_D$ is defined in an unambiguous way thanks to item (2) of proposition 3.2. A key property of the foliation $\mathcal{F}_D$ is described in the following

Corollary 3.1. *If $D \neq 0$ is a S^1-flat line bundle over a complex manifold M then the foliation $\mathcal{F}_D$ leaves D and every compact complex hypersurface contained in the complement of the support of D invariant.*

Let M be the surface describe in example 3.1 and D be a non-zero divisor on M supported on the quotient of one of the axis. Since the quotients of the two axis do not intersect each other it follows from corollary 3.1 that D is not S^1-flat, although D is $\mathbb{C}^*$-flat.

Another consequence of proposition 3.2 is a particular case of Theorem 2. We will include it here since the arguments are simpler then in the general case.

Corollary 3.2. *Let M be a compact complex manifold such that $2h^1(M, \mathbb{R}) = h^1(M, \mathcal{O}_M)$. If $\{H_i\}_{i \in \mathbb{N}}$ is an infinite set of pairwise disjoint hypersurfaces of M then there exists a holomorphic map $\rho : M \to C$, C a smooth algebraic curve, with connected fibers such that every H_i is a component of a fiber of ρ.*

Proof. Since $H^2(M, \mathbb{R}) = H^2(M, \mathbb{Q}) \otimes \mathbb{R}$ is finite dimensional there exists integers $k, n_1, \ldots, n_k$, $k > 0$, such that

$$c_{\mathbb{R}} \left(\mathcal{O}_M (\sum_{r=1}^{k} n_r H_r) \right) = 0 \,,$$

i.e., the divisor $D = \sum_{r=1}^{k} n_r H_r$ has zero real Chern class. From proposition 2.1(items (3) and (5)) we obtain that the line bundle $\mathcal{O}_M(\sum_{i=1}^{k} n_r H_r)$ is S^1-flat. From corollary 3.1 there exists a codimension one holomorphic foliation $\mathcal{F}$ of M leaving every H_i, $i \in \mathbb{N}$, invariant.

We now make use of of Ghys' version of Jouanolou's Theorem, see [3], to obtain a meromorphic first integral $g : M \dashrightarrow P^1_{\mathbb{C}}$ of $\mathcal{F}$. Since the hypersurfaces are $\{H_i\}$ are pairwise disjoint we can easily verify that the indeterminacy locus of g is empty and therefore g is holomorphic. ¿From Stein's factorization Theorem there exists an algebraic curve C, a fibration with $\rho : M \to C$ and a ramified covering $\pi : C \to P^1_{\mathbb{C}}$ such that $g = \pi \circ \rho$. $\square$

We end this section with examples of S^1-flat divisors over projective manifolds which are not associated to divisors linearly equivalent to zero.

Example 3.2. Let M be projective manifold and suppose that there exists a nontrivial homomorphism $\phi : \pi_1(M) \to S^1$. If $\pi : \tilde{M} \to M$ is the universal covering of M then we consider the codimension one foliation $\tilde{\mathcal{G}}$ over $M \times \mathbb{C}^2$ defined by the 1-form $\omega = xdy - ydx$ where (x, y) are the coordinates of $\mathbb{C}^2$. The homomorphism ϕ induces and action Φ of $\pi_1(M)$ on $\tilde{M} \times \mathbb{C}^2$ given by

$$\begin{aligned} \Phi : \pi_1(M) \times (\tilde{M} \times \mathbb{C}^2) &\to \tilde{M} \times \mathbb{C}^2 \\ (g, (p, (x, y))) &\mapsto (g \cdot p, (\phi(g)x, \phi(g)^{-1}y)) \end{aligned}$$

It is easy to see that the action Φ preserves the foliation $\tilde{\mathcal{G}}$.

The quotient of $\tilde{M} \times \mathbb{C}^2$ by Φ defines a rank 2 vector bundle E over M equipped with a codimension one foliation $\mathcal{G}$. Observe that $E = \mathcal{L} \oplus \mathcal{L}^*$ where $\mathcal{L}$ and $\mathcal{L}^*$ are flat line-bundles.

By GAGA's principle this vector bundle is algebraic and therefore $\mathbb{P}(E)$, the projectivization of E, is a projective variety. Note that $\mathbb{P}(E)$ carries two sections M_1 and M_2 corresponding to the split $E = \mathcal{L} \oplus \mathcal{L}^*$.

The foliation $\mathcal{G}$ induces a smooth codimension one foliation $\mathcal{F}$ of $\mathbb{P}(E)$ which leaves the two sections M_1 and M_2 of $\mathbb{P}(E)$ invariant. The divisor $D = M_1 - M_2$ is S^1-flat and it is possible to prove that $\mathcal{F} = \mathcal{F}_D$. Moreover we have that

(1) if the image of $\pi_1(M)$ is a non-trivial finite subgroup of S^1 then D is not linearly equivalent to zero but there exist a multiple of D linearly equivalent to zero;

(2) if the image of $\pi_1(M)$ is not a finite subgroup of S^1 then D, or any of its multiples, is not linearly equivalent to zero.

The next example is a variant of an example presented in [9] and attributed to Brendan Hassett.

Example 3.3. Let M be a projective variety with $h^1(M, \mathcal{O}_M) > 0$ and $\mathcal{L}$ be a non-trivial line-bundle with trivial Chern class, i.e., $\mathcal{L} \in Pic_0(M)$. Let D_1 be an effective divisor such that $h^0(M, \mathcal{O}_M(D_1) \otimes \mathcal{L}) > 0$. If D_2 is the zero divisor of a section of $\mathcal{O}_M(D_1) \otimes \mathcal{L}$ then $D = D_1 - D_2$ is S^1-flat. Moreover

(1) if $\mathcal{L}$ is a torsion element of $Pic_0(M)$ then D is not linearly equivalent to zero but there exist a multiple of D linearly equivalent to zero;

(2) if $\mathcal{L}$ is a non-torsion element of $Pic_0(M)$ then D, or any of its multiples, is not linearly equivalent to zero.

4. The Group $\Gamma(M)$ for compact complex manifolds

If M is a compact complex manifold let $\Gamma(M)$ be the group defined by

$$\Gamma(M) = \frac{\mathrm{Div}(M) \otimes \mathbb{Q}}{\mathrm{Div}_{S^1}(M) \otimes \mathbb{Q}}.$$

Since flat divisors have zero rational Chern class then there exists a rational Chern class map

$$c_{\mathbb{Q}} : \Gamma(M) \to H^2(M, \mathbb{Q}).$$

It follows from proposition 2.1 that for compact complex manifolds M with $h^1(M, \mathcal{O}_M) = 2h^2(M, \mathbb{R})$ that the map $c_{\mathbb{Q}}$ is injective. For projective manifolds it is a trivial matter to verify that this injectivity identifies the image of $\Gamma(M)$ with the rational Neron-Severi group of M.

In order to prove that $\Gamma(M)$ is finite diminensional for general compact complex manifolds we will consider the algebraic reduction of M.

4.1. **Divisors and Algebraic Reduction.** For the results mentioned on the next two paragraphs the reader should consult [10, pages 24–27] and references there within.

If M is a compact complex manifold then the field of meromorphic functions of M, denoted by $k(M)$, is a finitely generated extension of $\mathbb{C}$ whose transcendence degree is bounded by the dimension of M. The transcendence degree of M is called its algebraic dimension and will be denoted by $a(M)$. In the case $a(M) = \dim M$ then M is called a Moishezon manifold and there exists a finite succession of blow-ups along non-singular centers such that the resulting manifold is projective.

In general there exists a compact complex variety $\tilde{M}$, a bimeromorphic morphism $\psi : \tilde{M} \to M$ and a morphism $\pi : \tilde{M} \to N$ with connected fibers such that N is a smooth projective variety and

$$\psi^* k(M) = \pi^* k(N).$$

The projective variety N is called an *algebraic reduction* of M. Note that an algebraic reduction of M is unique up to bimeromorphic equivalence.

We will say that a hypersurface H of a complex variety M is *special* if, in the notations above, the restriction of $\pi \circ \psi^{-1}$ to H is a dominant meromorphic map, i.e., has dense image. Remark that a Moishezon variety does not have special hypersurfaces and every hypersurface of a variety of zero algebraic dimension is special.

The proposition below is a generalization of Theorem 6.2 of [1, page 129].

Proposition 4.1. *If M is a compact complex variety then there are at most $h^1(M, \Omega^1_M) + \dim M - a(M)$ special hypersurfaces.*

Proof. Suppose that H_k, $1 \le k \le h^1(M, \Omega^1_M) + \dim M - a(M) + 1$, are distinct special hypersurfaces of M and let $\mathbb{H} = \oplus_k \mathbb{C} \cdot H_k$ be the $\mathbb{C}$-vector space generated by them.

As in [3] we can define a morphism from $\theta : \mathbb{H} \to H^1(M, \Omega^1_M)$ as follows: for every H_k we can consider the associated line bundle $\mathcal{O}_M(H_k)$ and map it to $d \log \phi_{ij}$, where ϕ_{ij} are the transition functions of $\mathcal{O}_M(H_k)$; the morphism is them defined through linearity. If $\sum \lambda_k H_k$ belongs to the kernel of θ then we can define a global meromorphic 1-form with simple poles of residue λ_k along H_k.

¿From our assumptions we have that the dimension of the kernel of θ is at least $\dim M - a(M) + 1$ and we can therefore construct $\omega_1, \dots, \omega_l$, $l = \dim M - a(M) + 1$, meromorphic 1-forms over M such that the polar set of ω_r is

$$(\omega_r)_\infty = H_1 \cup H_2 \cup \ldots \cup H_h \cup H_{h+r} \, ,$$

where $h = h^1(M, \Omega^1_M)$. In particular the 1-forms ω_i are linearly independent over $\mathbb{C}$ and moreover since H_i are special theirs restriction to F, the closure of a general fiber of $\pi \circ \psi^{-1} : M \dashrightarrow N$, are still linearly independent over $\mathbb{C}$.

Let $\eta_1, \dots, \eta_{a(M)}$ be the pullback under $\pi \circ \psi^{-1}$ of rational 1-forms of N linearly independent over $k(N)$. Since we have now $\dim M + 1$ meromorphic 1-forms there exists a relation of the form

$$\sum_{i=1}^{l} f_i \omega_i = \sum_{j=1}^{a(M)} g_j \eta_j \, ,$$

where f_i and g_j are meromorphic functions of M.

If we take the restriction of the relation above to F then since every meromorphic function of M is constant along F we obtain that the restriction of the 1-forms ω_i to F are linearly dependent over $\mathbb{C}$. A contradiction which proves that there are at most $h^1(M, \Omega^1_M) + \dim M - a(M)$ special hypersurfaces on M. $\qquad \square$

The statement of Theorem 6.2 of [1, page 129] is the specialization to the case of surfaces of the following

Corollary 4.1. *If M is a compact complex variety of algebraic dimension zero then M has at most $h^1(M, \Omega^1_M) + \dim M$ hypersurfaces.*

Example 4.1. If M is the quotient of $\mathbb{C}^n \setminus \{0\}$ by a sufficiently general contraction then $a(M) = h^1(M, \Omega^1_M) = 0$ and M has $n = \dim M$ special hypersurfaces; this shows that the bound presented in proposition 4.1 above is sharp in every dimension.

4.2. **Proof Theorem 1.** If the algebraic dimension of M is zero then Theorem 1 is an immediate consequences of corollary 4.1.

If M is a Mosheizon variety then, as we have already mentioned, there exists a projective variety $\tilde{M}$ and a bimeromorphic morphism $\psi : \tilde{M} \to M$. Therefore the finitude of $\dim_{\mathbb{Q}} \Gamma(\tilde{M})$ implies the finitude of $\dim_{\mathbb{Q}} \Gamma(M)$.

It remains to deal with the cases where the algebraic dimension of M satisfies $0 < a(M) < \dim M$.

Without loss of generality we can suppose that there exists a holomorphic map from M to a smooth projective variety N with connected fibers. Let $\pi : M \to N$ be such map and set $\mathcal{R} \subset M$ as

$$\mathcal{R} = \{x \in M; \operatorname{rank} d\pi(x) < \dim N\} \, .$$

We will make use of the following lemma.

Lemma 4.1. *In the notations above if H be an irreducible hypersurface of M then*

(1) *if $\dim \pi(H) < \dim N - 1$ then $H \subset \mathcal{R}$;*
(2) *if $\pi(H) \not\subset \pi(\mathcal{R})$ and $\dim \pi(H) = \dim N - 1$ then, in the group of divisors of M, $\pi^*(\pi(H)) = H + E$, where E is an effective divisor supported on $\mathcal{R}$.*

Proof. Item (1) follows from the local form of submersions and item (2) follows from Sard's Theorem and the connectedness of the fibers of π. We leave the details to the reader. $\qquad\square$

Let now $S(M)$ denote the subgroup of $\mathrm{Div}(M)$ generated by the special divisors of M and $R(M)$ denote the subgroup of $\mathrm{Div(M)}$ generated by divisors supported on $\mathcal{R}$. Note that $S(M)$ and $R(M)$ have both of finite rank and the map

$$\mathrm{Div}(N) \oplus S(M) \oplus R(M) \quad \to \quad \mathrm{Div}(M)$$
$$(D, S, R) \quad \mapsto \quad \pi^*D + S + R$$

is surjective. Since π^* sends S^1-flat divisors of N to S^1-flat divisors of M and $\Gamma(N)$ is finite dimensional then Theorem 1 follows. $\qquad\square$

5. A key property of the foliation $\mathcal{F}_D$

We start this section with a simple lemma.

Lemma 5.1. *If D is a divisor of a compact complex manifold M then there exists a compact complex manifold $\tilde{M}$ and a bimeromorphic map $\pi : \tilde{M} \to M$ such that π^*D admits a decomposition of the form $\pi^*D = D_+ - D_-$ where D_+ and D_- are effective divisors with disjoint supports.*

Proof. ¿From Hironaka's desingularization Theorem we can suppose that the support of D has only normal crossing singularities and its irreducible components are smooth.

Write D as $P_0 - N_0$ where P_0 and N_0 are effective divisors without irreducible components in common in theirs supports. If $\mathcal{V}_0$ be the set of codimension two subvarieties V of M such that $V \subset H_P \cap H_N$ where H_P is an irreducible component of P_0 and H_N is an irreducible component of N_0. Since we are supposing that the support of D has normal crossings then every $V \in \mathcal{V}_0$ is smooth.

The proof follows from an induction on the cardinality of $\mathcal{V}_0$. We leave the details to the reader. $\qquad\square$

¿From now on we will say that *a divisor D is without base points* if $D = D_+ - D_-$ where D_+ and D_- are effective divisors with disjoint supports.

The next proposition is the cornerstone of the proofs of Theorems 2 and 5. It is in fact a generalization of item (1) of proposition 3.2.

Proposition 5.1. *Let $\mathcal{F}$ be a holomorphic foliation of a compact complex manifold M and D be a S^1-flat divisor of M. If $\mathcal{F}$ admits a transcendental leaf which do not intersect the support of D then $\mathcal{F} = \mathcal{F}_D$.*

Proof. ¿From lemma 5.1 we can suppose, without loss of generality, that the S^1-flat divisor D is without base points, i.e., $D = D_+ - D_-$ where D_+ and D_- are effective divisors with disjoint supports.

By definition the foliation $\mathcal{F}_D$ is induced by a closed meromorphic one form ω_D with polar set supported on D and admits a real first integral $F : M \to [0, \infty]$ such

that $F^{-1}(0)$ is equal to the support of D_+ and $F^{-1}(\infty)$ is equal to the support of D_-, see the proof of proposition 3.2 and definition 3.2.

Consider

$$F_{|L} : L \to [0, \infty]$$

the restriction of F to the transcendental leaf L of $\mathcal{F}$. Since F is locally defined as the modulus of a holomorphic function then $F_{|L}$ is either constant or an open map. If $F_{|L}$ is constant then L is a leaf of both $\mathcal{F}$ and $\mathcal{F}_D$. In particular the tangency locus of $\mathcal{F}$ and $\mathcal{F}_D$ contains the analytic closure L. Since L is not contained in any hypersurface then $\mathcal{F} = \mathcal{F}_D$.

From now on we will suppose that $F_{|L}$ is an open map. Recall that F can be locally written as

$$F = \left| \exp \left(\int \omega_D \right) \right|$$

and that all the periods of ω_D are purely imaginary complex numbers. If all the periods of ω_D are commensurable with $\pi\sqrt{-1}$ then there exists a positive integer n such that F^n is equal to the modulus of the complex function $f = \exp(n(\int \omega_D))$. Let $\overline{L}$ be the topological closure of L and $\partial L = \overline{L} \setminus L$. Since f is open and $\overline{L}$ does not intersect the support of D then $f(\partial L) = \overline{f(L)} \setminus f(L)$. Therefore ∂L is invariant by both $\mathcal{F}$ and $\mathcal{F}_D$. Since $f(L)$ is an open set relatively compact in $\mathbb{C}^* \subset \mathbb{P}^1$ we have that $\partial f(L)$ is infinite. In particular $\mathcal{F}$ and $\mathcal{F}_D$ have an infinite number of leaves in common. This is sufficient to show that $\mathcal{F} = \mathcal{F}_D$.

It remains to analyze the case where ω_D has at least two linearly independent periods. When this is the case the multi-valued function $f = \exp(\int \omega_D)$ has a monodromy group dense in S^1. Let $\pi : \tilde{M} \to M$ the covering of M associated to f and consider the commutative diagram below.

$$
\begin{array}{ccccc}
\pi^{-1}(L) & \longrightarrow & \tilde{M} & \xrightarrow{\ f\ } & \mathbb{P}^1 \\
\pi \downarrow & & \pi \downarrow & & \downarrow |\cdot| \\
L & \longrightarrow & M & \xrightarrow{\ F\ } & [0, \infty]
\end{array}
$$

Since $F_{|L}$ is open and $\overline{L}$ does not intersect the support of D then there exist positive real numbers N_- and N_+ such that $F(L) = (N_-, N_+)$. The density of the monodromy group of f in S^1 implies that

$$f(\pi^{-1}(L)) = \{z \in \mathbb{C}; N_- < |z| < N_+\}.$$

It follows easily that ∂L contains an infinite number of leaves of $\mathcal{F}$. This is sufficient to show that $\mathcal{F} = \mathcal{F}_D$.

$\square$

6. Compact Complex Varieties which fiber over a curve

The main purpose of this section is to prove Theorem 2. The proof will be based on the following lemma which is a corollary to proposition 5.1.

Lemma 6.1. *Let D_1 and D_2 be two S^1-flat divisors on a compact complex manifold M. If there exists a connected component of the support of D_1 which does not intersect the support of D_2 then $\mathcal{F}_{D_1} = \mathcal{F}_{D_2}$.*

Proof. Let ω_{D_1} and ω_{D_2} be the meromorphic 1-forms canonically associated to D_1 and D_2 and let V be the connected component of the support of D_1 which does not intersect the support of D_2. In the proof of proposition 5.1 we saw that $\mathcal{F}_{D_1}$ admits a real analytic first integral $F : M \to [0, \infty]$ and V is a level of F. Thus there exists an open neighborhood U of V in M which is saturated by the foliation $\mathcal{F}_{D_1}$, i.e., every leaf of $\mathcal{F}_{D_1}$ which intersects U is in fact contained in U. Moreover we can choose U such that $D_2 \cap U = \emptyset$.

Let L be an arbitrary leaf of $\mathcal{F}_{D_1}$ contained in $U \setminus V$. If for every such L, $\overline{L}$ is a complex subvariety of M then $\overline{L}$ is invariant by $\mathcal{F}_{D_2}$ by item (1) of proposition 3.2. Thus the restriction of $\mathcal{F}_{D_1}$ and $\mathcal{F}_{D_2}$ to U coincide and therefore $\mathcal{F}_{D_1} = \mathcal{F}_{D_2}$. Otherwise F_{D_1} has a transcendental leaf which does not intersect the support of D_2 and the lemma follows from proposition 5.1. $\qquad\square$

6.1. **A foliation theoretic proof of Totaro's Theorem.** ¿From lemma 6.1 we obtain a slight generalization of part of a result by Totaro [9, Theorem 2.1]. The original result is stated for complex projective manifolds and its original proof has the advantage to work over fields of positive characteristic, see [9, section 6]. Here we will adopt a foliation theoretic approach which works uniformly for every compact complex variety(and give no hint how to proceed to establish the Theorem in positive characteristic).

6.1.1. **Proof of Theorem 3.** From the hypothesis of the Theorem there exists integers $n_{12}, n_{23}, m_{12}, m_{23}$ such that

$$D_{12} = n_{12}D_1 - m_{12}D_2 \text{ and } D_{23} = n_{23}D_2 - m_{23}D_3$$

are S^1-flat divisors without base points. Lemma 6.1 implies that $\mathcal{F}_{D_{12}} = \mathcal{F}_{D_{23}}$.

We can now conclude as in the proof of Jouanolou's Theorem. Note that there exists a non-constant meromorphic function $F \in k(M)$ such that $\omega_{D_{12}} = F \cdot \omega_{D_{23}}$. Differentiating we obtain that $dF \wedge \omega_{D_{23}} = 0$, i.e., F is a meromorphic first integral of $\mathcal{F}_{D_{12}} = \mathcal{F}_{D_{23}}$. Since D_{12} is without base points it follows that F is in fact a holomorphic map $F : M \to \mathbb{P}^1$. The Theorem follows taking the Stein factorization of F. $\qquad\square$

6.2. **Proof of Theorem 2.** Considering the natural map $\mathcal{H} \otimes \mathbb{Q} \to \Gamma(M)$ it follows that there exists two S^1-flat divisors D and D' with support contained in $\mathcal{H}$ and such that there exists a component H of the support of D not contained in D'. At this point we can use lemma 6.1 to guarantee that $\mathcal{F}_D = \mathcal{F}_{D'}$ and conclude as in Jouanolou's Theorem, cf. the proof of Theorem 3. $\qquad\square$

7. Diffeomorphism type of Smooth Divisors

This section is devoted to the proof of Theorem 4. Before proceeding to the proof we would like to recall some remarks and examples made by Totaro in [9].

(1) If M is a projective variety with $H^1(M, \mathbb{R}) = 0$ then the Betti numbers and Hodge numbers of a smooth divisor are determined by its Chern class, see [9, remark 2]. More generally the same argument used there apply to any compact complex variety with $H^1(M, \mathcal{O}_M) = 0$.

(2) If M is a projective and D is an ample smooth divisor of M then the Betti and Hodge numbers of D are determined by its Chern class, see [9, remark 1].

(3) There exists smooth complex projective manifolds with two disjoint homologous smooth divisors which are both connected but have different Betti numbers.

7.1. **Proof of Theorem 4.** Let D_1 and D_2 be two connected divisors whose Γ-classes lie in a line of $\Gamma(M)$. There exists integers p and q such that $D = pD_1 - qD_2$ is S^1-flat. Thus we can choose a covering $\mathcal{U} = \{U_i\}$ os M and local rational functions f_i defining $D = pD_1 - qD_2$ such that the $f_i = f_{ij}f_j$ and f_{ij} are locally constant functions of modulus 1. As before we have that $\omega = \frac{df_i}{f_i}$ is a well-defined global meromorphic 1-form and moreover $F = |f_i| : M \to [0, \infty]$ is a well-defined continuous function(real-analytic outside the support of D).

We can also define a global (real) 1-form θ over $M \setminus (D_1 \cup D_2)$ by the relation $\theta_{|U_i} = d \arg(f_i)$, where arg denotes the complex argument, i.e., $f = |f| \cdot \exp(\arg(f))$. In a neighborhood of D_1 and D_2 the 1-form θ has mild algebraic singularities; if U is a sufficiently small neighborhood of a point $p \in D_1 \cup D_2$ then, over U, the foliation induced by θ is diffeomorphically equivalent to the foliation of $\Sigma \times (D \cap U)$ induced by $xdy - ydx$, where (x, y) are local real coordinates of a transversal Σ of D.

Integration along closed paths defines a homomorphism

$$\int \theta : H_1(M \setminus (D_1 \cup D_2), \mathbb{R}) \quad \to \quad \mathbb{R}$$

$$\gamma \quad \mapsto \quad \int_\gamma \theta$$

which sends γ_1 and γ_2, small loops around D_1 and D_2 respectively, to real numbers commensurable to π.

The inclusion of $M \setminus (D_1 \cup D_2)$ into M induces a surjective homomorphism $H_1(M \setminus (D_1 \cup D_2), \mathbb{R}) \to H_1(M, \mathbb{R})$ whose kernel is contained in the subspace of $H_1(M \setminus (D_1 \cup D_2), \mathbb{R})$ generated by γ_1 and γ_2. Therefore we can choose a morphism $T : H_1(M, \mathbb{R}) \to \mathbb{R}$ such that $(T + \int \theta)(\gamma)$ is commensurable to π for every $\gamma \in H_1(M \setminus (D_1 \cup D_2), \mathbb{Z})$. From DeRham's isomorphism we deduce the existence of a (real) closed 1-form η on M such that $\int_\gamma \theta + \eta$ is a rational multiple of π for every $\gamma \in H_1(M \setminus (D_1 \cup D_2), \mathbb{Z})$. Since $H_1(M \setminus (D_1 \cup D_2), \mathbb{Z})$ is finitely generated there exists an integer N such that

$$G = \exp\left(iN \int \theta + \eta\right) : M \setminus (D_1 \cup D_2) \to S^1$$

is a well-defined C^∞ function.

Since the local structure, around points of D_1, of the foliation induced by $\eta + \theta$ is the same of the foliation induced by θ if we take a smooth fiber $\tilde{D}_1$ of

$$F^N \times G : M \setminus (D_1 \cup D_2) \to (0, \infty) \times S^1 \cong \mathbb{C}^*$$

sufficiently close to D_1 then there exists a positive integer m such that $\tilde{D}_1$ is an étale $\mathbb{Z}/m$-covering of D_1. In an analogous way a smooth fiber $\tilde{D}_2$ of $F \times G$ sufficiently close to D_2 is an étale $\mathbb{Z}/n$-covering of D_2 for some positive integer n.

We claim that choosing η small enough we can join $q_1 = (F^N \times G)(\tilde{D}_1)$ to $q_2 = (F^N \times G)(\tilde{D}_2)$ by a differentiable path $\gamma : [0, 1] \to \mathbb{C}^*$ avoiding the critical values of $F^N \times G$. In fact if

$$\rho : \overline{M} \to M \setminus (D_1 \cup D_2)$$

denotes the universal covering of $M \setminus (D_1 \cup D_2)$ then $\rho^*\theta$ is exact and holomorphic. Thus $\rho^*(N\omega)$ admits a holomorphic primitive

$$\overline{G} : \overline{M} \to \mathbb{C}$$

which has a set of critical values of real codimension two. Thus if η is small enough then $\rho^*(F^N \times G)$ is sufficiently close to $\overline{G}$ and thus the set of critical values are also close. This is sufficient to prove the claim.

We can therefore lift the real vector field $\frac{\partial}{\partial t}$ on $[0,1]$ to $(F^N \times G)^{-1}(\gamma([0,1]))$. The flow of this vector field induces a diffeomorphism between $\tilde{D}_1$ and $\tilde{D}_2$. $\qquad\square$

Remark 1. Totaro shows in [9] that if D_1 and D_2 are smooth divisors on a projective manifold with the same Chern class then, after blowing up the intersection scheme of D_1 with D_2 and resolving the resulting variety, the strict transforms of D_1 and D_2 have the same Chern class. A similar argument shows that our result holds for smooth divisors D_1 and D_2, not necessarily disjoint, with the **same** Γ-class.

Note that a similar reasoning for smooth divisors which are not disjoint and whose Γ-classes lie in a line does not work. For instance if we take the a line L and a smooth cubic C on $\mathbb{P}^2$ they have non-diffeomorphic universal coverings and $3L - C$ is a S^1-flat divisor.

8. The Closure of Transcendental Leaves

8.1. **Hypersurfaces with ample normal bundle.** The original motivation of this work was to find an analogous of the following well-known fact for general projective varieties: if $\mathcal{F}$ is a holomorphic foliation of $\mathbb{P}^n$ and L is a transcendental leaf of $\mathcal{F}$ it is well known that the topological closure of L does intersect every compact hypersurface of $\mathbb{P}^n$, see [4].

The key point on the proof of the fact above is that the complement of any compact hypersurface of $\mathbb{P}^n$ is Stein, and even more it is in fact affine.

A first result, and almost obvious, result on this direction is the following

Proposition 8.1. *Let $\mathcal{F}$ be a holomorphic foliation of a projective variety M and H an effective divisor of M with ample normal bundle.* [1] *If L is a leaf of $\mathcal{F}$ then L is contained in contractible subvariety of M or the topological closure of L intersects H.*

Proof. Since the normal bundle of H is ample according to [5, Theorem 4.2, p. 110] there exists a positive integer k such that:

- the linear system $H^0(M, \mathcal{O}_M(nH))$ is free from base points;
- the natural map $\phi : M \to \mathbb{P}H^0(M, \mathcal{O}_M(nH))$ is holomorphic
- ϕ is biholomorphic on a neighborhood of H;
- the set $\phi(H)$ can be identified with the intersection of an hyperplane of $\mathbb{P}H^0(M, \mathcal{O}_M(nH))$ with $\phi(M)$, the image of ϕ.

Thus we can identify $\phi(M \setminus H)$ with an affine closed set of some affine space $\mathbb{C}^N$.

Suppose that L is not contracted by ϕ. Therefore we can choose a principal open subset U of $\mathbb{C}^N$ satisfying:

- $U \cap \phi(L) \neq \emptyset$;
- $\phi(M \setminus H) \cap U$ is a smooth affine variety;

---

[1]See [5] for a precise definition of divisors with ample normal bundle.

- $\phi_* \mathcal{F}$ restricted to $U \cap \phi(M \setminus H)$ is generated by a global section of the tangent sheaf of $U \cap \phi(M \setminus H)$.

Denote by $\tilde{M}$ the intersection of $\phi(M \setminus H)$ with U.

Let Θ_U be the sheaf of vector fields of U and $\Theta_{U,\tilde{M}}$ be the subsheaf formed by vector fields tangent to $\tilde{M}$. From the exact sequence of $\mathcal{O}_U$-coherent sheaves

$$0 \to \Theta_{U,\tilde{M}} \otimes I_{\tilde{M}} \to \Theta_{U,\tilde{M}} \to \Theta_{\tilde{M}} \to 0,$$

and the fact that coherent sheaves over affine varieties have no higher order cohomology we deduce that there exists a foliation of U, which naturally extends to a foliation $\mathcal{G}$ of $\mathbb{P}\mathrm{H}^0(V, \mathcal{O}_V(nH))$, whose restriction to $\tilde{M}$ coincides with $\phi_*(\mathcal{F})$.

Therefore let $p \in \phi(L) \cap \tilde{M}$ be a non-singular point of $\phi_*(\mathcal{F})$. The proposition follows if the closure of the leaf of $\mathcal{G}$ through p is not contained on any compact subset of $\mathbb{C}^N$. But this is precisely the case since no holomorphic vector field on $\mathbb{C}^N$ has a bounded leaf. $\qquad \square$

In the two dimensional case the above proposition specializes to

Corollary 8.1. *Let $\mathcal{F}$ be a holomorphic foliation of a projective surface S and D an effective divisor of S. If L is a transcendental leaf of $\mathcal{F}$ such that topological closure of L does not intersect the support of D then $D^2 \leq 0$.*

8.2. Proof of Theorem 5. Let $\mathcal{H}$ be the set of compact complex hypersurfaces of M which do not intersect the leaves of $\mathcal{F}$. Denote by Γ the natural map

$$\Gamma : \mathcal{H} \otimes \mathbb{Q} \to \Gamma(M).$$

If $\dim_{\mathbb{Q}} \ker \Gamma \geq 2$ then it follows that there exists two there exists two S^1-flat divisors D_1 and D_2 such that the supports of D_1 and D_2 are distinct. From proposition 5.1 we have that $\mathcal{F} = \mathcal{F}_{D_1} = \mathcal{F}_{D_2}$. Moreover, see argument in the proof of the first part of Theorem 5, $\mathcal{F}$ admits a meromorphic first integral and do not admit transcendental leaves. This shows that $\dim_{\mathbb{Q}} \ker \Gamma \leq 1$. In particular the cardinality of $\mathcal{H}$ is at most $\dim_{\mathbb{Q}} \Gamma(M) + 1$.

When the cardinality of $\mathcal{H}$ is precisely $\dim_{\mathbb{Q}} \Gamma(M) + 1$ we have that $\dim_{\mathbb{Q}} \ker \Gamma = 1$. If D is a generator of the kernel of Γ it follows from proposition 5.1 that $\mathcal{F} = \mathcal{F}_D$. This sufficient to prove (1).

If M is projective then we claim that Γ is not surjective. Otherwise there exists an ample divisor Z with support contained in $\mathcal{H}$ contradicting proposition 8.1. Moreover, since M is projective, if $h^1(M, \mathcal{O}_M) = 0$ then numerical and linear equivalence coincides modulo torsion. Thus arguing as above we prove (2). $\qquad \square$

We conclude by remarking that we cannot replace in the statement of item (1) of Theorem 5 the group $\Gamma(M)$ by the Neron-Severi group of M.

For instance if M is an arbitrary primary Hopf surface, i.e. S is the quotient of $\mathbb{C}^{*2}$ by a linear diagonal contraction, and E_1 and E_2 denote the elliptic curves on S obtained as the quotients of the coordinates axis then the divisor $D = E_1 - E_2$ is S^1-flat as the reader can easily verify. If the algebraic dimension of S is zero then every leaf of the foliation $\mathcal{F}_D$ distinct from E_1 and E_2 is transcendental and its topological closure does not intersect the support of D. Therefore for every transcendental leaf of $\mathcal{F}_D$ the set $\mathcal{H}$ has cardinality two while the Neron-Severi group has dimension zero, i.e., is the trivial group.

Appendix by Laurent Meersseman

It is a natural question to ask if the results of this paper are still true if we replace $\Gamma(M)$ by the Neron-Severi group of M in the statements of the Theorems. The aim of this appendix is to give a negative answer to this question, at least for Theorems 2,3 and 4. In other words, the group $\Gamma(M)$ is really the good object to consider in these problems.

Theorem 6. *There exists a compact, complex* 5-*manifold* N *with three pairwise disjoint smooth hypersurfaces* H_1, H_2 *and* H_3 *such that*

    (a) *The Neron-Severi group of* N *is reduced to zero.*
    (b) *The manifold* N *does not admit a holomorphic map onto a smooth curve.*
    (c) *The universal coverings of* H_1 *and* H_3 *are not homotopically equivalent.*

The example of the Theorem comes from the family of compact, complex manifolds constructed and studied in [7] as a generalization of [6]. Let us first recall very briefly this construction. Let $n > 2m$ be positive integers. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ be a configuration of n vectors of $\mathbb{C}^m$. Assume it is *admissible*, i.e. that it satisfies
- the Siegel condition: $0 \in \mathbb{C}^m$ belongs to the (real) convex hull of $(\Lambda_1, \ldots, \Lambda_n)$.
- the weak hyperbolicity condition: if 0 belongs to the convex hull of a subset of $(\Lambda_1, \ldots, \Lambda_n)$, then this subset has cardinal strictly greater than $2m$.

Consider the holomorphic foliation $\mathcal{F}$ of the projective space $\mathbb{P}^{n-1}$ given by the following action

$$(T, [z]) \in \mathbb{C}^m \times \mathbb{P}^{n-1} \longmapsto [\exp\langle\Lambda_1, T\rangle \cdot z_1, \ldots, \exp\langle\Lambda_n, T\rangle \cdot z_n] \in \mathbb{P}^{n-1}$$

where the brackets denote the homogeneous coordinates in $\mathbb{P}^{n-1}$ and where $\langle -, - \rangle$ is the *inner* product of $\mathbb{C}^n$. Define

$$\mathcal{N}_\Lambda = \{[z] \in \mathbb{P}^{n-1} \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0\}$$

which is a smooth manifold due to the weak hyperbolicity condition. Then, there exists an open dense subset $V \subset \mathbb{P}^{n-1}$ such that the restriction of $\mathcal{F}$ to V is regular and admits $\mathcal{N}_\Lambda$ as a global smooth transverse. Therefore, $\mathcal{N}_\Lambda$ can be endowed with a structure of (compact) complex manifold as leaf space of $\mathcal{F}$ restricted to V. We denote by N_Λ this compact complex manifold. It has dimension $n - m - 1$ and is not Kaehler if $n > 2m + 1$ (see [7], Theorem 2).

The standard action of the torus $(S^1)^n$ onto $\mathbb{C}^n$ leaves $\mathcal{N}_\Lambda$ invariant and the corresponding quotient space is easily seen to identify with a simple convex polytope (see [2], Lemma 0.11; simple means dual to a simplicial polytope). We denote by P_Λ the combinatorial type of this convex polytope. It has some remarkable properties:
(i) *Rigidity:* there is a 1 : 1 correspondence between the combinatorial classes of simple convex polytopes and the classes of manifolds N_Λ up to C^∞ equivariant diffeomorphism and up to product by circles, see [2], Theorem 4.1.
(ii) *Realization:* given any simple convex polytope P, there exists N_Λ such that $P_\Lambda = P$, see [7], Theorem 13.
(iii) *Submanifolds:* a codimension p face F of P_Λ corresponds to a codimension $2p$ holomorphic submanifold $N_{\Lambda'}$ of N_Λ such that $P_{\Lambda'} = F$, see [7], §V.

We are now in position to construct our example.

Proof. Consider the convex polyhedron P obtained from the cube by cutting off two adjacent vertices by a plane (cf [2], Example 11.5). It has two triangular

facets (corresponding to the vertices which were cut off), two rectangular ones, two pentagonal ones and finally two hexagonal ones. By (ii), there exists manifolds N_Λ such that $P_\Lambda = P$. Here N_Λ can moreover be assumed to be 2-connected (see [7], Theorem 13). Then, n is equal to 8, and m to 2 so that N_Λ has complex dimension 5. By (i), for all such choices of Λ, the C^∞-diffeomorphism type of N_Λ is the same. Fix such a Λ and set $N = N_\Lambda$. By Corollary 4.5 of [2], we may assume that Λ is generic (in the sense of condition (H) of [7], IV). The two triangular facets correspond by (iii) to two smooth hypersurfaces H_1 and H_2. Choose a rectangular facet of P, and let H_3 denote the corresponding hypersurface of N. Notice that H_1, H_2 and H_3 are pairwise disjoint since the corresponding facets are pairwise disjoint.

Since N is 2-connected, its Neron-Severi group is reduced to zero. This proves (a).

Since Λ is generic, by [7], Corollary of Theorem 4, then the manifold N does not have any non-constant meromorphic function. So cannot admit a holomorphic projection onto an algebraic curve. This proves (b).

Finally, using (i) and [7], VIII, we have that H_1 and H_2 are diffeomorphic to $S^5 \times (S^1)^3$, whereas H_3 is diffeomorphic to $S^3 \times S^3 \times (S^1)^2$. Now, the universal coverings of these two manifolds are not homotopy equivalent. This finishes the proof. $\square$ $\hfill\square$

Notice that Theorem 4 implies that the rank of $\Gamma(N)$ is greater than or equal to 2.

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Jorge Vitório Pereira
I.R.M.A.R. - Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex France
email: jvp@impa.br